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Cluster Dehn twists in cluster modular groups

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1 Introduction

The *cluster algebra*, introduced by Fomin-Zelevinsky [FZ02], is a commutative associative algebra associated with a (possibly weighted) quiver Q without loops and 2-cycles. It is a subalgebra of the field of rational functions $\mathbb{C}(A_1, \dots, A_N)$ obtained by applying a finite number of rational transformations called *cluster \mathcal{A} -transformations* to the initial variables A_1, \dots, A_N . The cluster algebra is effectively used to investigate the function algebras of the double Bruhat cells of a reductive algebraic group and solve the classical *total positivity problem*. Almost simultaneously, a geometric counter part (and a “dualization”) of the cluster algebra is formulated by Fock-Goncharov [FG09]. They associate a dual pair of schemes $\mathcal{A}_{|Q|}$ and $\mathcal{X}_{|Q|}$, each of which is equipped with a distinguished collection of birational toric charts whose transition functions are given by cluster \mathcal{A} - and \mathcal{X} -transformations. Here a cluster \mathcal{X} -transformation is another rational transformation. The pair $(\mathcal{A}_{|Q|}, \mathcal{X}_{|Q|})$ is called the *cluster ensemble* associated with Q . The cluster algebra lies in the function algebra of $\mathcal{A}_{|Q|}$.

The cluster transformations are induced by a *mutation*, which is a transformation of quivers. The *cluster modular group* $\Gamma_{|Q|}$ is, roughly speaking, the group of sequences of mutations which preserves a quiver. It acts on the cluster algebra and the cluster ensemble by compositions of cluster transformations.

Since the foundation, many fruitful connections between the cluster algebra/ensemble and a broad area of mathematics are found: higher Teichmüller theory [FG06], quantum groups [Ip16], integrable systems [GK13], and many others. In particular, the cluster modular group plays important roles in these theories: it gives a combinatorial description of the action of the mapping class group on higher Teichmüller spaces; the universal R-matrix of a quantum group is realized in the cluster modular group; it gives discrete flows in *cluster integrable systems*, and so on.

Example 1.1. A basic example is given by the quiver $Q = Q_\Delta$ associated with an ideal triangulation Δ of a marked surface Σ . In this case, the corresponding cluster

ensemble describes a combinatorial structure of Teichmüller spaces related to Σ : the positive real part $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ coincides with the *decorated Teichmüller space* introduced by Penner [Pen12] equipped with the lift of the Weil-Petersson form, $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ coincides with the *enhanced Teichmüller space* [FG07] equipped with the Goldman Poisson structure. They are two types of extensions of the Teichmüller space of Σ , which is the universal cover of the Riemann moduli space of Σ . The cluster modular group $\Gamma_{|Q|}$ contains the *mapping class group* $MC(\Sigma)$ as a subgroup of finite index [BS15]. Namely, each element $\phi \in MC(\Sigma)$ is represented by a mutation sequence and the actions on these Teichmüller spaces are represented by the corresponding composition of cluster transformations.

From this example, we would be able to say that: *the cluster ensemble is a generalization of the Teichmüller space, and the cluster modular group is a generalization of the mapping class group*. Then our general question is the following:

Problem 1.2. *Is it possible to generalize a theorem for the Teichmüller spaces/mapping class groups to a theorem for the cluster ensembles/cluster modular groups?*

In this manuscript, we introduce *cluster Dehn twists* in the cluster modular group and show that they have similar properties as Dehn twists in the mapping class groups of surfaces.

2 Definitions

In this section, we recall basic notions of cluster ensemble following [FG09]. For our purpose, it is enough to consider the positive real part of the cluster ensemble which is a pair of contractible manifolds, rather than considering the corresponding schemes.

A quiver without loops and 2cycles is given by the data $Q = (I, \varepsilon)$, where I is a finite set (the set of vertices), $\varepsilon = (\varepsilon_{ij})_{i,j \in I}$ is a skew-symmetric matrix (the entry ε_{ij} gives $\#\{\text{arrows } i \rightarrow j\} - \#\{\text{arrows } j \rightarrow i\}$). Let us consider a tuple $(Q, (A_i)_{i \in I}, (X_i)_{i \in I})$, called a *seed*, where Q is a quiver and $(A_i)_{i \in I}, (X_i)_{i \in I}$ are two bunches of commutative variables parametrized by the vertex set I of Q . For a vertex $k \in I$, we define the *mutation*

$\mu_k : (Q, (A_i)_{i \in I}, (X_i)_{i \in I}) \rightarrow (Q', (A'_i)_{i \in I}, (X'_i)_{i \in I})$ by the following formulas:

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & i = k \text{ or } j = k, \\ \varepsilon_{ij} + \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{otherwise,} \end{cases} \quad (1)$$

$$A'_i = \begin{cases} A_k^{-1} \left(\prod_{j: \varepsilon_{kj} > 0} A_j^{\varepsilon_{kj}} + \prod_{j: \varepsilon_{kj} < 0} A_j^{-\varepsilon_{kj}} \right) & i = k, \\ A_i & i \neq k. \end{cases} \quad (2)$$

$$X'_i = \begin{cases} X_k^{-1} & i = k, \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & i \neq k, \end{cases} \quad (3)$$

The *mutation class* $|Q|$ is the set of quivers which are obtained by finite sequences of mutations from Q . Henceforth, let us concentrate our attention to the \mathcal{A} -side. (The story for the \mathcal{X} -side goes similarly.) The partial data $(Q, (A_i)_{i \in I})$ of a seed is encoded in the following geometric object. Let \mathcal{A}_Q be the positive Euclidean space $\mathbb{R}_{>0}^I$ equipped with coordinates $(A_i)_{i \in I}$ and a presymplectic structure $\omega_Q := \sum_{i,j \in I} \varepsilon_{ij} d \log A_i \wedge d \log A_j$. The *cluster \mathcal{A} -transformation* is the isomorphism $\mu_k^a : (\mathcal{A}_Q, \omega_Q) \rightarrow (\mathcal{A}_{Q'}, \omega_{Q'})$ such that the pull-back $(\mu_k^a)^* A'_i$ is given by the right-hand side of the formula (2).

The cluster \mathcal{A} -space $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ is defined to be a presymplectic manifold which satisfies the following conditions.

1. For each quiver $Q' \in |Q|$, we have an isomorphism $A_{Q'} : \mathcal{A}_{|Q|}(\mathbb{R}_{>0}) \rightarrow (\mathcal{A}_{Q'}, \omega_{Q'})$.
2. If $Q'' = \mu_k(Q')$, then the corresponding coordinate transformation $A_{Q''} \circ A_{Q'}^{-1}$ coincides with the cluster transformation $\mu_k^a : (\mathcal{A}_{Q'}, \omega_{Q'}) \rightarrow (\mathcal{A}_{Q''}, \omega_{Q''})$.

It depends only on the mutation class $|Q|$ and it has the following natural symmetry group. A *mutation sequence* is a finite sequence ϕ of mutations and permutations on the set I . Let ϕ^a denote the corresponding composition of μ_k^a 's and σ^a 's. Here σ^a denotes the permutation of the coordinates corresponding to a permutation σ . We identify two mutation sequences ϕ_1 and ϕ_2 if $\phi_1^a = \phi_2^a$. The *cluster modular group* Γ_Q at Q is the group of mutation sequences from Q to Q , modulo this identification. If $Q' = \mu_k(Q)$, then the conjugation by μ_k gives a group isomorphism $\Gamma_{Q'} \cong \Gamma_Q$. Therefore we identify these groups via this isomorphism and denote the resulting abstract group by $\Gamma_{|Q|}$. When we fix a “basepoint” $Q' \in |Q|$, an element $\phi \in \Gamma_{|Q|}$ is represented by a mutation sequence. The action of $\Gamma_{|Q|}$ on $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ is given by the natural action $\Gamma_{Q'} \rightarrow \text{Aut}(\mathcal{A}_{Q'}, \omega_{Q'})$, $\phi \mapsto \phi^a$ for some $Q' \in |Q|$.

Remark 2.1. One can similarly define the cluster \mathcal{X} -space $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ using the cluster \mathcal{X} -transformation and a Poisson structure $\Pi_Q := \sum_{i,j \in I} \varepsilon_{ij} X_i \frac{\partial}{\partial X_i} \wedge X_j \frac{\partial}{\partial X_j}$ instead of ω_Q .

For each mutation sequence ϕ , one can associate the corresponding composition ϕ^x . It is known [Man14] that $\phi_1^a = \phi_2^a$ if and only if $\phi_1^x = \phi_2^x$. Hence the cluster modular group $\Gamma_{|Q|}$ acts on $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ as well.

Definition 2.2. An element $\phi \in \Gamma_{|Q|}$ of infinite order is called a *cluster Dehn twist* if there exist integers $n, l \in \mathbb{Z}$, $l \neq 0$, such that $\phi^n = ((i\ j)\mu_j)^l$ for some vertices $i, j \in I$ of a quiver $Q' \in |Q|$. Here $(i\ j)$ denotes the transposition of i and j .

Example 2.3 (Dehn twists and half-twists). Dehn twists and half-twists in the mapping class group of a marked surface are cluster Dehn twists. (Details are in the talk)

3 Parabolic dynamics

The cluster \mathcal{A} -space $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ admits a natural compactification $\overline{\mathcal{A}}_{|Q|}$, which we call the *Fock-Goncharov compactification*. It is constructed by attaching the *projectivized tropical space* at infinity. Crucial properties are: $\overline{\mathcal{A}}_{|Q|}$ is homeomorphic to a closed ball B^I and the action of the cluster modular group extends to $\overline{\mathcal{A}}_{|Q|}$ continuously. Then the action of a cluster Dehn twist on $\overline{\mathcal{A}}_{|Q|}$ has the following dynamical property:

Theorem 3.1. *Let Q be a connected quiver with at least 3 vertices, $\phi \in \Gamma_{|Q|}$ be a cluster Dehn twist. Then there exists a unique boundary point $\ell \in \partial\overline{\mathcal{A}}_{|Q|}$ such that*

$$\lim_{n \rightarrow \infty} \phi^{\pm n}(g) = \ell$$

in $\overline{\mathcal{A}}_{|Q|}$, for all $g \in \mathcal{A}_{|Q|}(\mathbb{R}_{>0})$.

In other words, the action of ϕ has parabolic dynamics. Note that the Dehn twist $t_C \in MC(\Sigma)$ along a simple closed curve $C \in \Sigma$ has parabolic dynamics in the Thurston compactification of the Teichmüller space: indeed, the limit point ℓ is given by the element represented by C in the Thurston boundary, which is the space of measured geodesic laminations on Σ .

Remark 3.2 (Nielsen-Thurston classification). In [Ish17], the elements of a cluster modular groups are classified into 3 types: periodic/cluster-reducible/cluster-pA (pseudo-Anosov). They are characterized by fixed point properties of the action on the Fock-Goncharov compactification $\overline{\mathcal{X}}_{|Q|}$ of the \mathcal{X} -space, modulo some technical conjectures on cluster algebras. In these terms, a cluster Dehn twist is a cluster-reducible element which is maximally reducible (cluster-reduction induces a 2-dimensional dynamics).

4 Generation of a cluster modular group by cluster Dehn twists

It is known that the mapping class group of a marked surface is virtually generated by Dehn twists and half-twists. Next we provide a generalization of this theorem. A quiver Q is of *finite mutation type* if the set $|Q|$ is finite. Q is of *finite type* if it is mutation-equivalent to a Dynkin quiver. It is known that the cluster modular group $\Gamma_{|Q|}$ is a finite group if Q is of finite type, and it is an infinite group if Q is of mutation finite type and not finite type.

Theorem 4.1 (Felikson-Shapiro-Tumarkin [FST12]). *Suppose a quiver Q is of finite mutation type and not finite type. Then either Q is associated with an ideal triangulation of a marked surface or mutation-equivalent to one of the following 8 quivers: $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.*

Then we prove the following.

Theorem 4.2. *Let Q be a quiver of finite mutation type. If Q is not of type $\tilde{E}_6^{(1,1)}, \tilde{E}_7^{(1,1)}$ or $\tilde{E}_8^{(1,1)}$, then the cluster modular group $\Gamma_{|Q|}$ is generated by cluster Dehn twists.*

We conjecture that for all quiver of finite mutation type, the corresponding cluster modular group is generated by cluster Dehn twists.

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